國立政治大學 **110** 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	組合學	考試日期	**2021** 年 **9** 月 **27** 日	考試時間	**09:00** 至 **12:00**

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **7** 個問題, 總計 **120** 分。

1. (16 %) Find the number of spanning trees in K_n (the complete graph of n vertices). If possible, use as many different methods as you can to derive your results.

2. (16 %) A *Hadamard matrix* of order n is an $n \times n$ matrix H with entries $+1$ and -1, such that $HH^T = nI$.
   1. Prove that if H is a Hadamard matrix of order n, then $n = 1, 2$, or $n \equiv 0 \bmod 4$.
   2. Construct a Hadamard matrix of order 16.

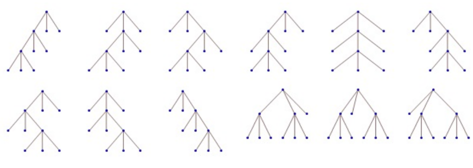
3. (16 %)
   1. Let a_n be the number of partitions of n with all parts ≥ 2. Express a_n in terms of $p(n)$, the number of partitions of n.
   2. For nonnegative integers n, r, compute $\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n$.

4. (16 %)
   1. If a simple graph of n vertices has no K_p, at most how many edges can it have?
   2. The sets $A_1, A_2, \ldots, A_k$ are distinct subsets of $\{1, 2, \ldots, n\}$ with $A_i \cap A_j \neq \emptyset$ for all i, j. Find the maximal possible value of k.

5. (16 %) Let G be a connected planar graph and e, v be the number of edges and vertices, respectively.
   1. If $e > 1$, prove that $e \leq 3v - 6$.
   2. If further, suppose all cycles of G are of length at least k. Prove that $e \leq \frac{k}{k-2}(v-2)$.

6. (20 %)
   1. Write down the definitions of the Stirling numbers of the (signless) first kind $c_{n,k}$ and the second kind $S_{n,k}$. Also $s_{k,m} := (-1)^{n-k}c_{n,k}$.
   2. Deduce recurrence formulas respectively for $c_{n,k}$ and $S_{n,k}$.
   3. Prove that $\sum_{k=m}^{n} S_{n,k}s_{k,m} = \delta_{n,m}$.

7. (20 %) A ternary tree is a rooted tree data structure in which each *inner node* has exactly three child nodes (distinguished as left, mid, right respectively). A node is a *leaf* if it has no child. It is clear that a ternary tree of n inner nodes has exactly $2n + 1$ leaves. Let t_n be the number of ternary trees with n inner nodes. For example, there are $t_3 = 12$ ternary trees of 3 inner nodes (and 7 leaves).



Find the functional equation for the generating function $T = \sum_{k=0}^{\infty} t_k z^k$, then derive both the exactly formula t_n and an asymptotic formula for t_n. (If you find this problem too hard, do the 'binary tree' case.)

命題老師簽章： **(Teacher's Signature)**	日期： **(Date)**	年 月 日	■ 試題隨卷繳交 ■ 不可使用計算機

命題紙使用說明： 試題將用原件印製，敬請使用黑色墨水正楷書寫或打字(紅色不能製版請勿使用)。
Remarks ： For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.