國立政治大學 111 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	組合學	考試日期	2022 年 9 月 19 日	考試時間	09:00 至 12:00

注意事項
- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **9** 個問題, 總計 **100** 分。

1. (20 %) Let $T_{n,p}$ denote the Turán graph on n vertices which is a complete multipartite graph with p partite sets $S_1, S_2, \ldots, S_p$ which are as nearly equal in size as possible, that is $||S_i| - |S_j|| \le 1$ for any i, j. Let K_{p+1} denote a complete graph on $p+1$ vertices.
(1) **Prove** that the number of edges in the Turán's graph $T_{n,p}$ is
$$\frac{(p-1)(n^2-r^2)}{2p} + \binom{r}{2},$$
provided that $n = tp + r$, where t and r are nonnegative integers, and $0 \le r < p$.
(2) **Prove** that if a simple graph G on n vertices with m edges does not contain K_{p+1} as a subgraph, then
$$m \le \frac{p-1}{2p} n^2$$

2. (10 %) A doubly stochastic matrix is a nonnegative square matrix M such that the sum of entries in each row of M is 1 and the sum of the entries in each column of M is also 1. A permutation matrix P is a doubly stochastic matrix whose entries are either 0 or 1. **Prove** that a doubly stochastic matrix M can be expressed as
$$M = c_1 P_1 + c_2 P_2 + \cdots + c_\ell P_\ell$$
where $P_1, \ldots, P_\ell$ are permutation matrices and $c_1, \ldots, c_\ell$ are positive real numbers such that $c_1 + \cdots + c_\ell = 1$.

3. (10 %) Let r, p, q be positive integers with $\min\{p, q\} \ge r \ge 1$. **Prove** that there exists **a minimal positive integer** $N(p, q; r)$ with the following property. Let S be a set with n elements. Suppose that all $\binom{n}{r}$ r-subsets of S are colored **red or blue**. Then if $n \ge N(p, q; r)$, we must have either some p-subset of S for which every r-subset is colored red or some q-subset of S in which every r-subset is colored blue.

4. (10 %) The Möbius function $\mu(d)$ is defined as
$$\mu(d) := \begin{cases} 1 & \text{if } d \text{ is the product of an even number of distinct primes,} \\ -1 & \text{if } d \text{ is the product of an odd number of distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$
The Riemann zeta function is defined in the complex plane with Re$(z) > 1$ such that
$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$
(1) **Prove** that $\sum_{d|n} \mu(d) = 0$ for $n > 1$.
(2) **Prove** that $\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$.

命題老師簽章： (Teacher's Signature)	日期： (Date) 2022 年 9 月 6 日	■ 試題隨卷繳交 ■ 不可使用計算機

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5. (10 %) Let $s(n,k)$ denote the **Stirling numbers of the first kind**.
(1) For $n \geq 0$, **prove** that

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n$$

where $(x)_n$ is defined as

$$(x)_n := \begin{cases} x(x-1)\cdots(x-n+1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

(2) **Prove** that

$$\sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!} = \frac{1}{k!}\left(\ln(1+z)\right)^k$$

6. (10 %) Let $x_1x_2\cdots x_n$ be the product of n numbers. Let u_n denote the number of ways of parenthesizing $x_1x_2\cdots x_n$ to specify the order of multiplication. For examples, $u_1 = 1$ and there are $u_4 = 5$ ways to parenthesize $x_1x_2x_3x_4$:

$$x_1(x_2(x_3x_4)),\ x_1((x_2x_3)x_4),\ (x_1x_2)(x_3x_4),\ (x_1(x_2x_3))x_4, \text{ and } ((x_1x_2)x_3)x_4$$

Prove that

$$u_n = \frac{1}{n}\binom{2n-2}{n-1}$$

7. (10 %) Define $p_{\text{odd}}(n)$ as the number of partitions of n into **odd parts** and $p_{\text{dist}}(n)$ as the number of partitions of n into **unequal parts**. For examples, we have $p_{\text{odd}}(6) = 4$ since

$$6 = (1+1+1+1+1+1) = (3+1+1+1) = (3+3) = (5+1)$$

and $p_{\text{dist}}(6) = 4$ since

$$6 = (6) = (5+1) = (4+2) = (3+2+1)$$

(1) **Prove** $p_{\text{odd}}(n) = p_{\text{dist}}(n)$ by using generating functions.
(2) **Prove** $p_{\text{odd}}(n) = p_{\text{dist}}(n)$ yet again, this time by constructing a bijection between partitions.

8. (10 %) Let P be a partially ordered finite set. Let m_P denote the minimum number of disjoint chains which together contain all elements of P, and let M_P denote the maximum number of elements in an antichain of P. **Prove** that $m_P = M_P$.

9. (10 %) Let P be a finite partially ordered set with the binary relation $\leq$. The zeta function ζ of P is defined by

$$\zeta(x,y) := \begin{cases} 1 & \text{if } x \leq y \text{ in } P \\ 0 & \text{otherwise} \end{cases}$$

which is a square matrix whose rows and columns are indexed by the elements of P. Let μ denote the Möbius function of P which is the inverse of ζ, that is $\mu\zeta = I$ (the identity matrix). If $x, y \in P$, a sequence $x = x_0 < x_1 < \cdots < x_k = y$ is called a chain of length k from x to y. Let $c_k(x,y)$ denote the number of such chains. For an example, if $x < y$, then $c_1(x,y) = 1$. **Prove** that

$$\mu(x,y) = \sum_{k\geq 1}(-1)^k c_k(x,y)$$

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