國立政治大學 112 學年度第一學期 博士班資格考 試題卷

**NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM**

| 系別 | 應用數學系 | 考試科目 | 組合學 | 考試日期 | 2023 年 9 月 18 日 | 考試時間 | 09:00 至 12:00 |
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注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **6** 個問題, 總計 **100** 分。

1. (a) (5 %) Let $A(G)$ be the adjacency matrix of a simple graph $G$. Show that the $i,j$-entry of $A^n$ is the number of walks of length $n$ on $G$ from $i$ to $j$.
   (b) (5 %) Let $P_n$ be the path graph on $n$ vertices and 1 one of its the endpoints. Find the $1,1$-entry of $A(P_{2n})^n$.
   (c) (5 %) Let $C_n$ be the cycle graph on $n$ vertices and 1 one of its vertices. Find the $1,1$-entry of $A(C_{2n})^n$.

2. Let $R(p,q)$ be the Ramsey number. That is, the smallest integer $k$ such that any blue-red coloring on the edges of $K_k$ always lead to a blue $K_p$ or a red $K_q$.
   (a) (10 %) Show that $R(p,q) \leq R(p-1,q) + R(p,q-1)$.
   (b) (5 %) Show that $R(p,q) \leq \binom{p+q-2}{p-1}$ for any $p,q \geq 2$.

3. (a) (10 %) A permutation $\sigma$ on $[n] = \{1,\ldots,n\}$ is called a derangement if $\sigma(i) \neq i$ for all $i \in [n]$. Use the principle of inclusion and exclusion to find the number $d_n$ of derangements on $[n]$.
   (b) (5 %) Find the probability of getting a derangement when $n$ is large. That is, find $\lim_{n\to\infty} \frac{d_n}{n!}$.

4. (15 %) Find the exponential generating function for the number of symmetric $n \times n$ permutation matrices.

5. (20 %) Let $\mathcal{B}_n$ be the boolean lattice. That is, the poset defined on the subsets of $\{1,\ldots,n\}$ with the subsets ordered by inclusion. Find the largest cardinality of an antichain in $\mathcal{B}_n$. Describe all the antichains achieving the maximum cardinality. Justify your answer.

6. (20 %) Let $\Gamma$ be a transportation flow with source $S$ and sink $T$ such that each edge has integral capacity. Show that the maximum flow equals the minimum cut. Moreover, there is a flow achieving the maximum such that its flow on each edge is integral.

林晉宏 2023 8 1