國立政治大學 111 學年度第二學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	高等機率論	考試日期	2023 年 2 月 20 日	考試時間	09:00 至 12:00

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 10 個問題，從其中挑六題做，並請清楚的註明哪六題要採計分數。採計分數的六題中，完整的答對四題保證通過。請書寫清楚，並嚴謹的解釋所有的步驟，任何書寫或是解釋上的不清楚都會導致沒有分數。

1. 第一題題目
Suppose that X is a random variable uniformly distributed on $[-1,1]$. Find $E[X|X^2]$. In order to get points, you have to justify your answer rigorously.

2. 第二題題目
Find a way to describe any σ-field on the set $\{1,2,...,n\}$ where n is a positive integer.

3. 第三題題目
Let $([0,1),\mathcal{B}([0,1)),\lambda)$ be the probability space where $\mathcal{B}([0,1))$ is Borel sets on $[0,1)$ and λ is Lebesgue measure on $[0,1)$. Define

$$X_n(\omega)=\sum_{k=1}^{2^n}\left(\frac{1-(-1)^k}{2}\right)1_{[\frac{k-1}{2^n},\frac{k}{2^n})}(\omega),\ n\geq 1.$$

Show that X_n's are independent.

4. 第四題題目
Suppose that $\{X_n,\ n\geq 1\}$ is a sequence of i.i.d. random variables with

$$P\{X_1=1\}=p \text{ and } P\{X_1=-1\}=1-p, \text{ where } \frac{1}{2}<p<1.$$

Define $S_n=\sum_{j=1}^{n}X_j$ for $n\geq 1$ and $S_0=0$, show that this random walk $\{S_n,\ n\geq 0\}$ is transient.

5. 第五題題目
Let $(\Omega,\ \mathcal{B},\ P)$ be the probability space, X be an integrable random variable and $\{\mathcal{F}_n\subseteq\mathcal{B},\ n\geq 1\}$ be a family of σ-fields. Show that

$$\{E[X|\mathcal{F}_n],\ n\geq 1\}$$

is uniformly integrable.

6. 第六題題目
Let $\{X_n,\ n\geq 1\}$ be a sequence of i.i.d. random variables with $P\{X_1=1\}=P\{X_1=-1\}=1/2$. Define

$$M_n=\sum_{k=1}^{n}a_kX_k,\ n\geq 1.$$

Show that if $\sum_{n=1}^{\infty}a_n^2<\infty$, then M_n converges a.s. as $n\to\infty$.

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7. 第七題題目
Suppose that X and Y are independent with common distribution having mean zero and variance one, and suppose further that

$$\frac{X+Y}{\sqrt{2}} \stackrel{d}{=} X \stackrel{d}{=} Y.$$

Show that both X and Y have a $N(0,1)$ distribution. Hint: Use characteristic function and the central limit theorem.

8. 第八題題目
Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence of random variables. Show that if for every $\epsilon > 0$

$$\sum_{n=1}^{\infty} P[|X_1| \geq \epsilon n] < \infty,$$

then

$$\lim_{n\to\infty} \left|\frac{X_n}{n}\right| = 0 \quad a.s.$$

9. 第九題題目
Prove or disprove that if X_n converges to X in distribution, then X_n converges to X a.s.

10. 第十題題目
Suppose that $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables such that

$$P[X_n = n^3] = P[X_n = -n^3] = \frac{1}{2n^2} \text{ and } P[X_n = 1] = P[X_n = -1] = \frac{1}{2} - \frac{1}{2n^2}.$$

Show that

$$\frac{\sum_{j=1}^{n} X_j}{n} \text{ converges to 0 a.s.}$$

as $n \to \infty$.