國立政治大學 114 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	高等機率論	考試日期	2025 年 9 月 8 日	考試時間	09:00-12:00

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 請書寫清楚，並嚴謹的解釋所有的步驟，任何書寫或是解釋上的不清楚都會導致沒有分數。
- 本試題卷共有 **10** 個問題，從其中挑 **6** 題做答，並請清楚的註明哪 **6** 題要採計分數。採計分數的 **6** 題中，每題 **16** 分，整張考券另外贈送 **4** 分。

1. Let $\{M_n,\, n \ge 1\}$ be a Markov chain on the state space $\{0, 1, 2, \cdots\}$ with the following transition probabilities:
$$p(k, k+1) = p \text{ and } p(k, k-1) = q, \text{ for } k \ge 1,$$
and $p(0, 1) = 1$, where $0 < p < q$ and $p + q = 1$. Find the probability distribution $\{\pi(k),\, k \ge 0\}$ such that
$$\pi(k-1)p(k-1, k) = \pi(k)p(k, k-1) \text{ for } k \ge 1.$$

2. Let $f(x)$ and $g(x)$ be two probability density functions defined on $\mathbb{R}$, where $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$. Show that
$$\int_{\mathbb{R}} \left(\log \frac{f(x)}{g(x)} \right) f(x)dx \ge 0,$$
provided that the integral is well-defined.

3. Let events A and B be given. Prove that A and B are independent if and only if the indicator functions
$$1_A \text{ and } 1_B \text{ are independent.}$$

4. For a given positive constant $c > 0$, define $W_t := B_{ct}/\sqrt{c}$ for $t \ge 0$, where $\{B_t,\, t \ge 0\}$ is a Brownian motion. Show that $\{W_t,\, t \ge 0\}$ is also a Brownian motion.

5. Let $\{X_n,\, n \ge 1\}$ be a sequence of i.i.d. random variables. If
$$\lim_{n\to\infty} \left| \frac{X_n}{n} \right| = 0 \quad \text{a.s.},$$
then
$$\sum_{n=1}^{\infty} P\{|X_1| \ge \epsilon n\} < \infty,$$
for every $\epsilon > 0$.

命題老師簽章：(Teacher's Signature)	日期：(Date) 年 月 日	■ 試題隨卷繳交 ■ 不可使用計算機

系別	應用數學系	考試科目	高等機率論	考試日期	2025年9月8日	考試時間	09:00-12:00

6. Let X be a random variable with finite mean, and let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{F}$ be σ-fields such that $\mathcal{G}_1 \subseteq \mathcal{F} \subseteq \mathcal{G}_2$. Show that if $E[X|\mathcal{G}_1] = E[X|\mathcal{G}_2]$ a.s., then

$$E[X|\mathcal{F}] = E[X|\mathcal{G}_1] \text{ a.s.}$$

7. Let Y_s be a random variable that follows a Poisson distribution with parameter $s > 0$. The probability mass function of Y_s is given by

$$P[Y_s = k] = e^{-s}\frac{s^k}{k!} \text{ for } k \geq 0.$$

Prove that

$$\frac{Y_s - s}{\sqrt{s}} \text{ converges weakly to } N(0,1),$$

as $s \to \infty$, where $N(0,1)$ denotes a normal random variable with mean zero and variance one.

8. Let $\{X_n,\, n \geq 1\}$ be a uniformly integrable sequence that converges to 0 in probability as $n \to \infty$. Show that $\{X_n,\, n \geq 1\}$ converges to 0 in L_1 as $n \to \infty$.

9. Suppose that $\{X_n,\, n \geq 1\}$ are i.i.d. random variables with mean zero and variance σ^2. Define the random walk as follows:

$$S_n = \sum_{k=1}^{n} X_k \text{ for } n \geq 1 \text{ and } S_0 = 0.$$

Let $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$ for $n \geq 1$ and $\mathcal{F}_0 = \emptyset$. Show that
(1) $\{(S_n,\, \mathcal{F}_n),\, n \geq 0\}$ is a martingale.
(2) $\{(S_n^2 - n\sigma^2,\, \mathcal{F}_n),\, n \geq 0\}$ is a martingale.

10. Let $\{X_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with mean zero and variance one. Define the sequence $\{Y_n,\, n \geq 1\}$ as follows:

$$Y_n = \begin{cases} X_n, & \text{if } |X_n| \leq n; \\ X_n^2, & \text{if } |X_n| > n, \end{cases}$$

for all $n \geq 1$. Show that

$$\frac{\sum_{k=1}^{n} Y_k}{\sqrt{n}}$$

converges weakly to $N(0,1)$ as $n \to \infty$, where $N(0,1)$ represents a normal random variable with mean zero and variance one.

命題老師簽章： (Teacher's Signature)	日期： 年 月 日 (Date)	■ 試題隨卷繳交 ■ 不可使用計算機