NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	微分方程式	考試日期	**2025 年 9 月 8 日**	考試時間	**09:00-12:00**

注意事項
- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **5** 個問題, 總計 **100** 分。

Please show all your work.

1. (20 %) Consider the problem

$$\frac{dx}{dt} = x^{\frac{1}{3}}, \quad x(0) = x_0\,.$$

   Prove that

   a. If $x_0 = 0$, then the solution is not unique for $t \in (-\infty, \infty)$.

   b. If $x_0 > 0$, then there exists small $\epsilon > 0$ such that the solution is unique for $t \in (-\epsilon, \epsilon)$.

2. (20 %) Let $\eta(t)$ be a nonnegative differentiable function on $[0, T]$ which satisfies the inequality

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t)\,,$$

   where $\phi(t)$ and $\psi(t)$ are nonnegative continuous functions on $[0, T]$. Prove that for all $0 \le t \le T$

$$\eta(t) \le e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]\,.$$

3. (20 %) Consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 - x_1(x_1^2 + x_2^2)\,, \\[2mm] \dfrac{dx_2}{dt} = -x_1 - x_2(x_1^2 + x_2^2)\,. \end{cases} \quad (N)$$

   a. Prove that $(0, 0)$ is asymptotically stable for the nonlinear system (N).

   b. Let $x' = Ax$ be the linearized system of (N) around $(0, 0)$, find the matrix A and prove that $(0, 0)$ is the center for the linearized system.

4. (20 %) Let $\Phi(x, x_0)$ be the principal fundamental matrix of the linear system $u' = A(x)u$ in an interval J. Prove that $\Phi(x, x_0) = \Phi(x, x_1)\Phi(x_1, x_0)$, where $x_1 \in J$.

5. (20 %) Let $p(t)$ and $q(t)$ be continuous functions in $[t_0, \infty)$. Suppose that all solutions of $x'' + p(t)x = 0$ are bounded in $[t_0, \infty)$. Show that all solutions of $x'' + (p(t) + q(t))\, x = 0$ are bounded in $[t_0, \infty)$ provided $\displaystyle\int_{t_0}^{\infty} |q(t)|\, dt < \infty$.

命題老師簽章： **(Teacher's Signature)**	日期：　　年　　月　　日 **(Date)**	■ 試題隨卷繳交 ■ 不可使用計算機