國立政治大學 115 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	微分方程式	考試日期	2026 年 9 月 14 日	考試時間	13:00-16:00

注意事項

- **務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。**
- **本試題卷共有 5 個問題, 總計 100 分。**

Please show all your work.

1. (15 %) Let $\alpha \in \mathbb{R}$ and define

$$A_\alpha = \begin{pmatrix} \alpha & 0 & 1 \\ 1 & \alpha - 1 & 2 \\ 1 & -1 & \alpha + 1 \end{pmatrix}.$$

Consider the linear system

$$\dot{\mathbf{x}}(t) = A_\alpha \mathbf{x}(t).$$

(1) (10 %) Find the general solution of the system.

(2) (5 %) Determine the values of α for which the zero solution is asymptotically stable.

2. (20 %) Consider the following initial value problem:

$$\begin{cases} \dot{x} = x(2 - x - y), \\ \dot{y} = y(1 + x - y), \\ x(0) = x_0 > 0, \qquad y(0) = y_0 > 0. \end{cases}$$

(1) (8%) Prove that there exists $\delta > 0$ such that the problem admits a unique solution $(x(t), y(t))$ on $[0, \delta)$, and that $x(t) > 0$, and $y(t) > 0$ as long as the solution exists.

(2) (8%) Prove that

$$x(t) + y(t) \leq \max\{x_0 + y_0, 4\},$$

as long as the solution exists.

(3) (4%) Can the solution be extended to all $t \geq 0$? Justify your answer.

3. (25 %) Consider the following planar autonomous system:

$$\begin{cases} \dot{x} = 2x - 2y - x\big(x^2 + xy + 2y^2\big), \\ \dot{y} = 2x + 2y - y\big(x^2 + xy + 2y^2\big). \end{cases} \qquad \textbf{(P)}$$

Answer the following questions:

(1) (5%) State the Poincaré–Bendixson theorem for a planar autonomous system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2.$$

(2) (12%) Construct a positively invariant annulus and use the Poincaré–Bendixson theorem to prove that **(P)** admits at least one nonconstant periodic orbit.

命題老師簽章： (Teacher's Signature)	日期： (Date)	年 月 日	■ 試題隨卷繳交 ■ 不可使用計算機

命題紙使用說明： 試題將用原件印製，敬請使用黑色墨水正楷書寫或打字 (紅色不能製版請勿使用)。
Remarks : For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.

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(3) (8%) Determine whether system **(P)** has a unique nonconstant periodic orbit. Justify your answer.

4. (20 %) Consider the following 2π-periodic linear system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & h(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad h(t) = \frac{\cos t + \sin t}{2 + \sin t - \cos t}.$$

(1) (10%) Find the principal fundamental matrix $\Phi(t)$ satisfying

$$\Phi(0) = I_2.$$

(2) (4%) Find the Floquet multipliers associated with the period 2π.

(3) (6%) Find a 2π-periodic nonsingular matrix-valued function $P : \mathbb{R} \to \mathbb{R}^{2\times 2}$ and a constant matrix $R \in \mathbb{R}^{2\times 2}$ such that

$$\Phi(t) = P(t)e^{tR}, \qquad t \in \mathbb{R}.$$

5. (20 %) Let $a < b$ and $q_1, q_2 \in C([a, b]; \mathbb{R})$ satisfy

$$q_1(t) \le q_2(t), \quad t \in [a, b]; \quad q_1 \not\equiv q_2 \quad \text{on } [a, b].$$

Suppose that u is a nontrivial solution of

$$u'' + q_1(t)u = 0$$

and that a and b are two consecutive zeros of u.

(1) (10%) Prove that every nontrivial solution v of

$$v'' + q_2(t)v = 0$$

has at least one zero in (a, b). You may use the identity

$$\big(u'v - uv'\big)' = (q_2 - q_1)uv.$$

(2) (10%) Prove that every nontrivial solution of

$$y'' + (1 + t^2)y = 0$$

has infinitely many zeros on $(0, \infty)$.

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