國立政治大學 **111** 學年度第二學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	微分方程式	考試日期	**2023** 年 **2** 月 **20** 日	考試時間	**13:00** 至 **16:00**

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **6** 個問題, 總計 **100** 分。

1. (10 %) Define

$$A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

2. (20 %) Consider the problem

$$\begin{cases} \dot{x} = x(1-x) - 2xy, \\ \dot{y} = y(x-3), \\ x(0) = x_0 > 0, \quad y(0) = y_0 > 0 \end{cases}$$

(1) (10 %) Prove that the problem has a unique solution for all $t > 0$.

(2) (10 %) Prove that there exists $C > 0$ such that $0 < x(t), y(t) \leq C$ for all $t > 0$.

3. (20 %) Consider the second-order linear equation

$$\ddot{x} + q(t)x = 0, \tag{1}$$

where q is continuous in $\mathbb{R}$. A solution $x(t)$ is said to be *oscillatory* if it has no last zero. That is, if $x(t_1) = 0$, then there exists an $t_2 > t_1$ such that $x(t_2) = 0$.

(1) (10 %) Suppose that $a, b \in \mathbb{R}$ are the consecutive zeros of a nontrivial solution $x(t)$ of (1) and let $r(t)$ be continuous with $r(t) \geq (\not\equiv) q(t)$ for $t \in [a, b]$. Prove that every nontrivial solution $y(t)$ of the equation

$$\ddot{y} + r(t)y = 0$$

has a zero in (a, b).

(2) (10 %) Prove that each solution of the equation $\ddot{z} + (c/t^2)z = 0$ for $t > 0$ has infinitely many positive zeros if $c > 1/4$, and only has finitely many positive zeros if $c < 1/4$.

4. (25 %) We consider a disease model where the number of individuals in the susceptible (healthy) class is represented by $S(t)$ and the number of individuals who are infectious (sick) is represented by $I(t)$. The following model accounts for the possibility of individuals becoming susceptible again after recovery from infection:

$$\begin{cases} \dot{S} = \Lambda - \beta SI - \mu S + \kappa I \\ \dot{I} = \beta SI - (\kappa + \mu + \alpha)I, \end{cases}$$

where $\Lambda, \beta, \mu, \kappa, \alpha$ are positive constants.

(1) (5 %) Find all equilibrium points in $\{(S, I) \,|\, S \geq 0,\ I \geq 0\}$.

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(2) (10 %) Let $E^0(S, I)$ be the disease-free equilibrium, which is represented as $(S_0, 0)$ for some constant $S_0 > 0$. Prove that there exits a critical value R_0 (depending on system parameters) such that E^0 is asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.

(3) (10 %) Construct a Lyapunov function $V(S, I)$ to prove that E^0 is globally asymptotically stable if $R_0 < 1$ (Hint: quadratic functions may be useful).

5. (15 %) Consider the system

$$\begin{cases} \dot{x} = x - y - x(x^2 + 5y^2), \\ \dot{y} = x + y - y(x^2 + y^2). \end{cases}$$

(1) (5 %) State the Poincaré-Bendixon Theorem for two-dimensional autonomous system.

(2) (10 %) Prove that there exists a periodic orbit in the trapping region $1/\sqrt{2} \leq r \leq 1$, where $r := \sqrt{x^2 + y^2}$ (Hint: rewrite the system in polar coordinates).

6. (10 %) Suppose that the $n \times n$ matrix $A(t)$ is continuous and periodic of period ω in $\mathbb{R}$. Also, let $\Psi(t)$ be a fundamental matrix of the differential system $\dot{\mathbf{x}} = A(t)\mathbf{x}$. Prove that the differential system has a nontrivial periodic solution $\mathbf{x}(t)$ of period ω if and only if $\det(\Psi(0) - \Psi(\omega)) = 0$.