| 考試科目 Course | 實變函數論 | 開課系級 Dept, & Class | 研究所 | 日　　期 Date, Period | 100 年 9 月 19 日 上午 9:00~12:00 | 試題編號 Course No. | |

本試卷共有 6 個題目，
碩士班：請選 5 題作答，每題 20 分，請在答案卷最前面註明所選的 5 題，否則依學生作答之前 5 題計分。
博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

1. Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Does the integral $\int_{-\infty}^{\infty} f(x)\, dx$ exist as an improper Riemann integral?

   (b) Is $f(x)$ Lebesgue integrable over $(-\infty, \infty)$?
   Prove your answer!

2. Let $p(x)$ be a polynomial function given by $p(x) = \sum_{i=1}^{k} a_i x^i$.
   (a) Show that for any positive y and natural number n, $|p(y/n)| \leq \sum_{i=1}^{k} |a_i| y^i$.
   (b) Show that

$$n \int_0^{\infty} p(x) e^{-nx} dx \to p(0) \quad \text{as } n \to \infty.$$

3. Let X be a normed linear space. Show that the set X^* of all bounded linear functionals on X is a Banach space.

4. Let $f \in L^1(\mathbb{R})$ be a uniformly continuous function on $\mathbb{R}$. Show that $\lim_{|x| \to \infty} f(x) = 0$.

5. Suppose that $\{f_n\}$ is a sequence of nonnegative integrable functions such that $f_n \to f$ a.e., with f integrable, and $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$. Prove that $\int_{\mathbb{R}} |f_n - f| \to 0$.

6. Suppose that $f \in L^1(\mathbb{R})$ is a absolutely continuous function on $\mathbb{R}$. Show that if in addition

$$\lim_{t \to 0+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| dx = 0$$

   Then $f \equiv 0$.

本考試：☑ 不需使用簡易計算機，□ 使用簡易計算機　　　←請出題老師勾選，謝謝！