（榮譽第一）

國立政治大學 應用數學系 106 學年度第一學期　學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	實變函數論	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 下午 14:00~17:00	試題編號 Course	

本試卷共有 6 個題目，

碩士班：請選 5 題作答，每題 20 分，請在答案卷最前面註明所選的 5 題，否則依學生作答之前 5 題計分。

博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

1. Show that every nonempty open set S in $\mathbb{R}$ is the union of a countable collection of disjoint component intervals of S and characterize all open connected subsets of $\mathbb{R}$. (Justify your answer)

2. (a) Let $\overline{B}(x,r)$ be a closed ball in $\mathbb{R}^n$ and $f: \overline{B}(x,r) \to \overline{B}(x,r)$ be a mapping satisfying $\|f(x')-f(x'')\| \le \frac{1}{2}\|x'-x''\|$ for all $x', x'' \in \overline{B}(x,r)$. Show that f has a unique fixed point; (b) show that all norms on $\mathbb{R}^n$ are equivalent.

3. (a) Let $f \in L^1(\mathbb{R})$. Show that the function $F(y) = \int_{\mathbb{R}} f(x+y)\,dm(x)$, $y \in \mathbb{R}$, is a well-defined continuous function on $\mathbb{R}$.
(b) In the space $C[0,1]$ of all continuous real-valued functions on $[0,1]$. Define, for $f \in C[0,1]$, $\|f\|_1 = \int_0^1 |f(x)|\,dx$. Show that $\|\cdot\|_1$ is a norm on $C[0,1]$. Is it a Banach space?

4. (a) Show that the function $\varphi(x) = -\log x$ is convex on $(0,\infty)$.
(b) Using Jensen's inequality to show that $\prod_{i=1}^{n} x_i^{\alpha_i} \le \sum_{i=1}^{n} \alpha_i x_i$, where $x_i \ge 0$, $\alpha_i > 0$, $1 \le i \le n$, and $\sum_{i=1}^{n} \alpha_i = 1$.

5. State and prove the Lebesgue dominated convergence theorem.

6. Let $\{x_n\}$ be a real sequence and $\sum_{n=1}^{\infty} x_n^4$ converges absolutely. Does $\sum_{n=1}^{\infty} \frac{x_n}{n}$ converges absolutely? (Justify your answer).

本考試：☑ 不需使用簡易計算機，☐ 使用簡易計算機　←請出題老師勾選，謝謝！

命題老師：(Teacher)	（簽章）(Signature & date) 106 年 9 月 9 日	試題隨卷繳交

命題紙使用說明：試題將用原件印製，敬請使用黑色墨水正楷書寫或打字（紅色不能製版請勿使用）。

Remarks ：For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.