

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試 科目	實變函數論	考試 日期	2022 年 9 月 19 日	考試 時間	13:00 至 16:00
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注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 7 個問題，總計 100 分。

- (10 %) Find a closed subset F of $\mathbb{R} \setminus \mathbb{Q}$ with a positive Lebesgue measure.
- (15 %) Let $X \neq \emptyset$, $\mathcal{A} \subseteq 2^X$ and $f : X \rightarrow \mathbb{R}$ be a function. Suppose that (X, \mathcal{A}, μ) is a measure space, f^2 is measurable and $X(f > 0)$ is measurable, where $X(f > 0) = \{x \in X : f(x) > 0\}$. Show that f is a measurable function.
- (15 %) Let A and B be non-empty sets. Prove that

- $\chi_{A \cap B} = \chi_A \cdot \chi_B$.
- $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.
- $\chi_{A \setminus B} = \chi_A - \chi_{A \cap B}$.

- (15 %) Let H be the Haar function defined by

$$H(x) := \begin{cases} 1, & x \in [0, 1/2) \\ -1, & x \in [1/2, 1) \end{cases}$$

Set

$$H_{jk}(x) := 2^{j/2} H(2^j x - k), \quad j, k \in \mathbb{Z}$$

and

$$\Psi := \{H_{j,k} : j, k \in \mathbb{Z}\}.$$

Show that Ψ is an orthonormal basis for $L^2(\mathbb{R})$.

- (15 %)
 - What is that a measure is translation invariant?
 - Let $A \subseteq \mathbb{R}$ be measurable $a \in \mathbb{R}$, $\delta > 0$. When $|x| < \delta$, there are at least one of $a + x$ and $a - x$ belonging to A . Show that $mA \geq \delta$, where mA denotes the Lebesgue measure of a measurable set A .
- (15 %) Suppose $1 < p < \infty$, $1/p + 1/q = 1$, and $f \in L^p(\mathbb{R})$. Prove the following equalities:

1.

$$\|f\|_p = \sup_{g \in L^q, \|g\|_q=1} \left| \int_{\mathbb{R}} f(x) \overline{g(x)} dx \right|.$$

2.

$$\|f\|_p^p = p \int_0^\infty t^{p-1} m(\mathbb{R}(|f| > t)) dt.$$

- (15 %) Let $M(X)$ denote the collection of all real-valued measurable functions defined on the measure space (X, \mathcal{A}, μ) . Suppose $g \in L^1(X)$, $f_n \in M(X)$ and $g \leq f_n$ ($n \in \mathbb{N}$). Show that

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X g d\mu.$$

命題老師簽章：

(Teacher's Signature)

日期：

(Date)

111 年 9 月 6 日

■ 試題隨卷繳交

■ 不可使用計算機

命題紙使用說明：試題將用原件印製，敬請使用黑色墨水正楷書寫或打字（紅色不能製版請勿使用）。

Remarks : For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.