

## NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試 科目	實變函數論	考試 日期	2025 年 2 月 24 日	考試 時間	09:00-12:00
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## 注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 7 個問題，總計 100 分。

1. (15 %) Let  $T : [0, 1) \rightarrow [0, 1)$  be a transformation on the Borel probability space  $([0, 1), \mathcal{B}, \mu)$  defined as

$$Tx = 2x \pmod{1}.$$

Show that  $T$  is measure-preserving; that is,  $\mu(T^{-1}A) = \mu(A)$  for all  $A \in \mathcal{B}$ .

2. (15 %) Let  $(x_{m,n})_{m,n \in \mathbb{N}}$  be a doubly infinite sequence of extended nonnegative real numbers. Then

$$\sum_{(m,n) \in \mathbb{N}^2} x_{m,n} = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} x_{m,n} = \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} x_{m,n}$$

3. (15 %) Let  $(X, \mathcal{B}, \mu)$  be a measure space, and let  $f, g : X \rightarrow [0, +\infty]$  be measurable. Then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

4. (10 %) (Prove or Disprove) Let  $(X, \mathcal{B}, \mu)$  be a measure space and  $T : X \rightarrow X$  is a measure-preserving transformation. Suppose  $A \in \mathcal{B}$  is of positive measure. Show that for almost every  $x \in A$ , there exists  $k \in \mathbb{N}$  such that  $T^k(x) \in A$ .

5. (15 %) Let  $(X, \mathcal{B}, \mu)$  be a measure space and  $f$  be measurable. Suppose  $\int_A f d\mu = 0$  for all  $A \in \mathcal{B}$ . Show that  $f = 0$  almost everywhere.

6. (15 %) Suppose  $f_k \rightarrow f$  in  $L^p$ ,  $1 \leq p < \infty$ ,  $g_k \rightarrow g$  pointwise, and  $\|g_k\|_\infty \leq M$  for all  $k$ . Show that  $f_k g_k \rightarrow f g$  in  $L^p$ .

7. (15 %) Let  $Z$  be a measure zero subset  $\mathbb{R}$ . Show that  $Z' = \{z^2 : z \in Z\}$  also has measure zero.

命題老師簽章： (Teacher's Signature)	日期： (Date)	年 月 日	<input type="checkbox"/> 試題隨卷繳交 <input type="checkbox"/> 不可使用計算機
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命題紙使用說明：試題將用原件印製，敬請使用黑色墨水正楷書寫或打字(紅色不能製版請勿使用)。

Remarks : For the convenience of reprinting please Write questions in black or blue-black ( but no red ) ink.