國立政治大學應用數學系九十一學年度第二學期學科考試試題

科目：實變函數論

1. Let $C[a,b]$ be the space of all continuous real valued functions defined on the compact interval $[a,b]$.

   (a) Is $C[a,b]$ a closed subspace of $L^\infty([a,b])$?

   (b) Is $C[a,b]$ a closed subspace of $L^1([a,b])$?

   (You should justify your answer!)

2. (a) Let M be a metric space and $\mathfrak{B}$ be a collection of pairwise disjoint open balls in M. Show that if M is separable, then $\mathfrak{B}$ is at most countable.

   (b) Discuss the separability of $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$.

3. State and prove the Riemann-Lebesgue Lemma.

4. Let X be a Banach space and F be a closed subspace of X.

   (a) Define the quotient space X/F.

   (b) Define the quotient norm on X/F.

   (c) Prove, under the quotient norm, X/F becomes a Banach space.

5. Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set with finite Lebesgue measure $\lambda_n(E)$. Suppose $f : E \longrightarrow \mathbb{R}^*$ is a Lebesgue measurable function and

$$E_k = \{\, x \in E \mid (k-1) \leq |\, f(x)\,| < k \,\}, k = 1, 2, 3, \ldots$$

   Show that, for $1 \leq p < \infty, f \in L^p(E) \Longleftrightarrow \sum_{k=1}^{\infty} k^p \lambda_n(E_k) < \infty$.

6. Let

$$\beta(x) = \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1 \end{cases}, x \in \mathbb{R}^n$$

   and

$$\alpha(x) = \beta(x) \left(\int_{\mathbb{R}^n} \beta(x)\, dx \right)^{-1}, x \in \mathbb{R}^n$$

$$\alpha_\epsilon(x) = \epsilon^{-n} \alpha(x/\epsilon), \epsilon > 0, x \in \mathbb{R}^n.$$

   Show that

   (a) $\alpha \in C_0^\infty(\mathbb{R}^n), supp\alpha = \bar{B}(0;1)$ and $\int_{\mathbb{R}^n} \alpha(x)\, dx = 1$.

   (b) $\alpha_\epsilon \in C_0^\infty(\mathbb{R}^n), supp\alpha_\epsilon = \bar{B}(0;\epsilon)$ and $\int_{\mathbb{R}^n} \alpha_\epsilon(x)\, dx = 1$.

   (c) If $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, then $f * \alpha_\epsilon \longrightarrow f$ in $L^p(\mathbb{R}^n)$ as $\epsilon \longrightarrow 0$. In particular, $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.