國立政治大學應用數學系九十二學年度第一學期研究生學科考試試題

科目：實變函數論

1. Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{A_n\}$ be a sequence in $\mathcal{F}$ with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that the set of all points belonging to infinite many A_n's has measure zero.

2. Let $f(x) = e^{-[x]}$, where $[x]$ is the greatest integer function. Is $f(x)$ integrable over $[0, \infty)$ with respect to the Lebesgue measure? If yes, compute its integral.

3. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, k be a bounded measurable function on X and $1 \leq p \leq \infty$. Define T on $L^p(X, \mathcal{F}, \mu)$ by
$$(Tf)(x) = k(x)f(x), \quad \forall\, x \in X.$$
Show that T is a bounded linear operator on $L^p(X, \mathcal{F}, \mu)$ and compute $\|T\|$ the operator norm of T.

4. Show that the space $C[a, b]$ of continuous functions on $[a, b]$ is not complete with respect to L^2-norm. What is the completion of $C[a, b]$ with respect to L^2-norm.

5. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\{f_n\}$ be a sequence of measurable functions on X. Discuss the relation of convergence a.e. on X, convergence in measure on X and L^p-convergence of $\{f_n\}$, $1 \leq p \leq \infty$.

6. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{I}, \nu)$ be two measure spaces and
$$f(x, y) = h(x)g(y), \quad \forall\, (x, y) \in X \times Y,$$
where $h \in L^1(X, \mathcal{F}, \mu)$ and $g \in L^1(Y, \mathcal{I}, \nu)$. Show that f is measurable on the product measurable space $(X \times Y, \mathcal{F} \times \mathcal{I})$ and
$$\int_{X \times Y} f\, d(\mu \times \nu) = \left(\int_X h\, d\mu \right) \left(\int_Y g\, d\nu \right).$$