國立政治大學應用數學系九十二學年度第二學期研究生學科考試試題

科目：實變函數論

1. Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on a finite-dimensional normed linear space X. Prove that they are equivalent.

2. Let $f \in L^1([0,1])$. Show that $x^n f(x) \in L^1([0,1])$ for all $n = 1, 2, \ldots$, and $\int_0^1 x^n f(x)\, dx \to 0$ as $n \to \infty$.

3. Let (X, S, μ) be a measure space. Identify the dual space $L^2(X, S, \mu)^*$ of $L^2(X, S, \mu)$. You must justify your answer!

4. Let $f \in L^1([a,b])$ and $F(x) = \int_a^x f(t)\, dt$, $x \in [a,b]$, be the indefinite integral of f on $[a,b]$. Show that F is continuous on $[a,b]$ and of bounded variation on $[a,b]$.

5. State and prove the Jensen's inequality for integral.

6. Let (X, S) be a measurable space and $\{\mu_n\}$ be a sequence of measures on (X, S) such that $\{\mu_n(E)\}$ is an increasing sequence for all $E \in S$. Show that the set function μ defined by $\mu(E) = \lim_{n\to\infty} \mu_n(E)$, $E \in S$, is a well-defined measure on (X, S).

7. Let X be a compact Hausdorff space and $C(X)$ be the space of real-valued continuous functions on X. Let $\|f\| = \max_{x\in X} |f(x)|$, $f \in C(X)$. Show that $(C(X), \|\cdot\|)$ is a Banach space.