國立政治大學應用數學系九十四學年度第二學期研究生學科考試試題

科目：實變函數論

1. Let (X, A, μ) be a measure space.

   (a) Suppose that $1 \leq p < r < \infty$. Prove that $L^p(X, A, \mu) \cap L^\infty(X, A, \mu) \subseteq L^r(X, A, \mu)$. Moreover, show that if $f \in L^p(X, A, \mu) \cap L^\infty(X, A, \mu)$, then $||f||_r \leq ||f||_p^{p/r} ||f||_\infty^{1-p/r}$.

   (b) If $f \in L^r(X, A, \mu) \cap L^\infty(X, A, \mu)$ for some $1 \leq r < \infty$, then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

2. Let $\Omega = \{(x, y) \in R^2 \mid 0 < x < \infty, 0 < y < 1\}$ and $f(x, y) = ye^{-xy} \sin x$. Show that

   (a) Ω is a Lebesgue measurable subset of R^2.

   (b) f is a Lebesgue measurable function on Ω.

   (c) Is $f(x, y)$ Lebesgue integrable on Ω? If yes, find the Lebesgue integral of $f(x, y)$ over Ω.

3. Let H be a Hilbert space and $B(H)$ be the collection of all bounded linear operators on H. Show that $B(H)$ is a Banach space with respect to the operator norm of linear operator.

4. For $1 \leq p \leq \infty$, is $L^p(X, A, \mu)$ a Hilbert space? (Justify your answer!)

5. Let λ and μ be σ-finite measures on a measurable space. If $\lambda << \mu$ and $f = d\lambda/d\mu$, then, for all nonnegative measurable functions g on X, we have

$$\int_X g \, d\lambda = \int_X gf \, d\mu.$$

6. Let (X, A, μ) be a measure space and $\{f_n\}$ be a sequence of integrable functions on X such that $f_n \to f$ a.e. on X. Show that if

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0,$$

   then

$$\lim_{n\to\infty} \int_X |f_n| \, d\mu = \int_X |f| \, d\mu.$$