國立政治大學應用數學系九十五學年度第一學期研究生學科考試試題

科目：實變函數論

1. Let $X = C[a,b]$ be the space of continuous functions on $[a,b]$, $||f||_\infty = \max_{a\le t\le b}|f(t)|$ and $||f||_1 = \int_a^b |f(t)|\,dt$, $f \in X$. Show that
   (a) $||\cdot||_\infty$ and $||f||_1$ are norms on X;
   (b) $(X, ||\cdot||_\infty)$ is a Banach space, but $(X, ||\cdot||_1)$ is not;
   (c) $\lim_{p\to\infty}(\int_a^b |f(t)|^p\,dt)^{1/p} = ||f||_\infty$, for all $f \in X$.
2. Let $1 < p < \infty$ and $f \in L^p[0,\infty)$. Show that, for all $x > 0$, $|\int_0^\infty e^{-tx} f(t)\,dt| \le ||f||_p (xq)^{1/q}$ for some $1 < q < \infty$.
3. Let (X, S, μ) be a measure space. What is the dual space of $L^p(X, S, \mu)$, $1 \le p \le \infty$?
4. State and prove the Lebesgue dominated convergence theorem.
5. Let (X, S, μ) be a measure space. Show that $f_n \xrightarrow{\mu} f$ as $n \to \infty$ if and only if
$$\lim_{n\to\infty} \int_X \frac{|f_n - f|}{1 + |f_n - f|}\,d\mu = 0,$$
   where $\mu(X) < \infty$.
6. Describe the way of constructing the Lebesgue measure λ_n on $\mathbb{R}^n$.