國立政治大學應用數學系九十五學年度第二學期研究生學科考試試題

科目：實變函數論

1. Let (X, S, μ) be a measure space and f be a nonnegative integrable function on X. Show that the set function ν defined by

$$\nu(E) = \int_E f\, d\mu,\ E \in S,$$

is a finite measure on (X, S) and $\nu << \mu$.

2. State and prove the Lebesgue dominated convergence theorem.

3. Let ν be a signed measure on a measurable space (X, S). Show that there exists a unique pair of measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

4. Let M be a compact metric space. Show that

   (a) Every continuous function $f : M \to R$ admits both maximum and minimum values.

   (b) Every closed subset of M is compact.

5. (a) What is complete measure space?

   (b) Show that every measure space admits a completion.

6. Let (X, S, μ) be a finite measure space and $f \in L^\infty(X, S, \mu)$.

   (a) Show that $f \in L^p(X, S, \mu)$ for all $1 \le p < \infty$ and

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

   (b) Evaluate

$$\lim_{p\to\infty} \left(\int_0^1 e^{-px^2}\, dx \right)^{1/p}$$

   if exists.