# 國立政治大學應用數學系九十六學年度第一學期研究生學科考試試題

科目：實變函數論

1. Define $(Tf)(x) = \frac{1}{x}\int_0^x f(y)\,dy$ and $(Sf)(x) = \int_x^\infty \frac{f(y)}{y}\,dy$. Show that $T$ and $S$ are bounded linear operators on $L^p(0,\infty)$ and $L^q(0,\infty)$, respectively, where $1 < p \le \infty$ and $1 \le q < \infty$. Moreover, $||T|| \le \frac{p}{p-1}$ if $1 < p < \infty$ and $||T|| \le 1$ if $p = \infty$; $||S|| \le q$.

2. Let $Y$ be a topological space and $X$ be a set. If $\{f_\alpha\}_{\alpha\in I}$ is a collection of mappings from $X$ into $Y$, can you define a topology on $X$ so that each $f_\alpha$, $\alpha \in I$, becomes a continuous mapping from $X$ into $Y$?

3. Let $X$ be a normed linear space and $N$ be a closed subspace of $X$. Describe the quotient space and the quotient norm on $X/N$; Moreover, if $X$ is a Banach space, then so is $X/N$ under quotient norm.

4. Let $(X, S, \mu)$ be a measure space, $1 \le p < \infty$ and $f \in L^p(X, S, \mu)$. Show that the set $\{x \in X \mid f(x) \ne 0\}$ is of $\sigma$-finite and $\mu(\{x \in X \mid |f(x)| \ge n\}) \to 0$ as $n \to \infty$.

5. Prove the following statements:

   (a) Let $f_n = n^{-1/p}\chi_{[0,n]}$, $n = 1, 2, \ldots$, where $1 \le p < \infty$. Show that $f_n \to 0$ uniformly on $\mathbb{R}$, but $f_n \nrightarrow 0$ in $L^p(\mathbb{R})$.

   (b) Let $g_n = \chi_{[n,n+1]}$, $n = 1, 2, \ldots$. Show that $g_n \to 0$ pointwise on $\mathbb{R}$, but $g_n \nrightarrow 0$ in measure.

6. Let $(X, S)$ be a measurable space and $\nu$ be a signed measure on $(X, S)$. Show that

   (a) If $\{A_n\}$ is a sequence of positive set w.r.t. $\nu$, then so is $\bigcup_{n=1}^\infty A_n$.

   (b) There exists a partition $\{A, B\}$ of $X$ with $A$ positive set and $B$ negative set w.r.t. $\nu$.

   (c) Describe the positive, negative and total variation of $\nu$.