國立政治大學 **114** 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	實變函數論	考試日期	**2025** 年 **9** 月 **8** 日	考試時間	**09:00-12:00**

注意事項: 本試題共有二頁

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **6** 個問題, 總計 **100** 分。

1. (a) (10 %) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $f'(x)$ exists and $|f'(x)| \leq 1$ for almost all $x \in \mathbb{R}$. Is it true that

$$f(b) - f(a) = \int_{[a,b]} f'(x)\, dx \qquad \text{for every closed intervals } [a, b] \subset \mathbb{R}?$$

Justify your answer.

(b) Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \in (0, \infty); \\ 1, & \text{if } x = 0. \end{cases}$

i. (5 %) $f \in L^1([0, \infty))$? Justify your answer.

ii. (5 %) Does the improper Riemann integral $\displaystyle\int_0^\infty f(x)\, dx$ converge? Justify your answer.

2. (15 %) Prove the integral version of Minkowski's inequality for $1 \leq p \leq \infty$, and measurable function $f(x, y)$:

$$\left[\int \left(\int |f(x, y)|\, dx\right)^p dy\right]^{1/p} \leq \int \left(\int |f(x, y)|^p\, dy\right)^{1/p} dx.$$

3. Let $E \subset \mathbb{R}^n$ be any subset, recalling that the outer measure is defined via

$$m_*(E) = \inf\{m(\mathcal{O}) : \ E \subset \mathcal{O}, \text{ and } \mathcal{O} \text{ is an open set in } \mathbb{R}^n\}.$$

One can also define an inner measure $m^*(E)$ by

$$m^*(E) = \sup\{m(F) : \ F \subset E, \text{ and } F \text{ is a closed set in } \mathbb{R}^n\}.$$

Show that

(a) (7 %) $m^*(E) \leq m_*(E)$.

(b) (8 %) E is measurable if and only if that $m^*(E) = m_*(E)$, provided that $m_*(E) < \infty$.

命題老師簽章：	日期： 年 月 日	■ 試題隨卷繳交
(Teacher's Signature)	**(Date)**	■ 不可使用計算機

命題紙使用說明： 試題將用原件印製，敬請使用黑色墨水正楷書寫或打字 (紅色不能製版請勿使用)。
Remarks : For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.

國立政治大學 **114** 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	實變函數論	考試日期	**2025** 年 **9** 月 **8** 日	考試時間	**09:00-12:00**

4. Let $E \subset \mathbb{R}^n$ be a measurable set, and we say that x is a point of Lebesgue density of E if

$$\lim_{m(B)\to 0,\ x\in B} \frac{m(B\cap E)}{m(B)} = 1,$$

where B is a ball center at x. Show that

(a) (7 %) Almost every point $x \in E$ is a point of density of E.

(b) (8 %) Almost every point $x \notin E$ is not a point of density of E.

5. (15 %) Given $0 < \epsilon < 1$. Construct a dense measurable subset $E \subset [0,1]$ such that the outer measure of E is ϵ.

6. (20 %) Let $f(x,y)$, $0 \le x, y \le 1$ satisfy the following conditions: For each x, $f(x,y)$ is an integrable function of y, and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Show that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

命題老師簽章：林奕亘 (Teacher's Signature)	日期：2025 年 8 月 29 日 (Date)	■ 試題隨卷繳交 ■ 不可使用計算機

命題紙使用說明： 試題將用原件印製，敬請使用黑色墨水正楷書寫或打字 (紅色不能製版請勿使用)。
Remarks： For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.