國立政治大學 111 學年度第二學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	實變函數論	考試日期	**2023 年 2 月 20 日**	考試時間	**09:00 至 12:00**

注意事項
- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **7** 個問題，總計 **100** 分。

1. (10 %) Let f be a measurable function and B a Borel set. Show that $f^{-1}(B)$ is a measurable set.

2. (15 %) Construct a sequence $\{f_k\}$ to show that convergence in measure does not imply pointwise convergence a.e., even for those functions defined on the sets of finite measure.

3. (15 %) Let $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & x \notin \mathbb{Q} \\ x^3 & x \in \mathbb{Q} \end{cases}.$$

   Prove that f is Lebesgue integrable on $[0.1]$ and evaluate $\displaystyle\int_0^1 f(x)dx$.

4. (15 %) Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}.$$

5. (15 %) Let $\phi(x), x \in \mathbb{R}^n$, be a bounbded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0}(f * \phi_\varepsilon)(x) = f(x)$$

   in the Lebesgue set of f. Here $(f * g)(x)$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$

6. (15 %) Let

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}dt, \qquad \text{for } x > 0 \text{ and } y > 0.$$

   Show that $\ln B(x, y)$ is a convex function of x, for each fixed y by Hölder's inequality.

7. (15 %) If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\| \leq M$ for all k, prove that $f_k g_k \to fg$ in L^p.