國立政治大學　**112** 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	實變函數論	考試日期	**2023** 年 **9** 月 **18** 日	考試時間	**13:00** 至 **16:00**

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **8** 個問題, 總計 **100** 分。

1. (10 %) Suppose that the function f is defined on a measurable set X and has the property that $X(f > c)$ is measurable for each rational number c. Is f measurable?

2. (15 %) Let $X \neq \varnothing$ and $\mathscr{A} \subseteq 2^X$. Suppose that $(X, \mathscr{A}, \mu)$ is a measure space.

   a. If there exists a measurable set A in $\mathscr{A}$ such that $\mu A < \infty$, show that $\mu\varnothing = 0$.

   b. Let $A, B \in \mathscr{A}$. Show that $\mu A + \mu B = \mu(A \bigcup B) + \mu(A \bigcap B)$.

   c. If $\{E_k\}_{k\in\mathbb{N}}$ be a collection of measurable subsets of X. Prove that

$$\mu\Big(\bigcup_k E_k\Big) \leq \sum_k \mu E_k.$$

3. (10 %) Suppose f^2 is measurable and $X(f > 0)$ is measurable. Show that f is a measurable function.

4. (15 %) Let ϕ be increasing and right-continuous, i.e., $\lim\limits_{y\downarrow x} \phi(y) = \phi(x)$. Define μ_ϕ^* by

$$\mu_\phi^*(E) := \inf\Big\{ \sum_n \big[\phi(b_n) - \phi(a_n)\big] : E \subseteq \bigcup_n (a_n, b_n)\Big\}, \quad E \subseteq \mathbb{R}.$$

   a. Prove that for each $b \in \mathbb{R}$, $\mu_\phi^*(\{b\}) = \phi(b) - \phi(b^-)$, where $\phi(b^-) = \lim\limits_{x\uparrow b} \alpha(x)$.

   b. Find the outer measure $m^*(C)$, where m^* is the Lebesgue outer measure and C is the Cantor set in $[0, 1]$.

   c. Find the outer measure $\mu_\rho^*(C)$, where ρ is the Cantor function.

5. (10 %) Let $E \subseteq [0, 1]$ be Lebesgue measurable and let $\varepsilon > 0$. Prove that there are finite number of open intervals, (a_i, b_i), $i = 1, 2, \cdots, n$ satisfying the following condition: if $G = \bigcup\limits_{i=1}^n (a_i, b_i)$, then $m(G \setminus E) < \varepsilon$.

6. (10 %) Let $f_n \in M(X)$ for $n \in \mathbb{N}$. If $f_n \geq g \in L^1$, then show $\int_X \liminf\limits_n f_n d\mu \leq \liminf\limits_n \int_X f_n d\mu$; if $f_n \leq g \in L^1$, then show $\int_X \limsup\limits_n f_n d\mu \geq \limsup\limits_n \int_X f_n d\mu$.

7. (10 %) Find the limit: $\lim\limits_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{[1 + (x/n)]^n} dx$.

8. (20 %) Suppose $1 < p < \infty$, $1/p + 1/q = 1$, and $f \in L^p(\mathbb{R})$. Prove the following equalities:

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a.
$$\|f\|_p = \sup_{g\in L^q, \|g\|_q=1} \left| \int_{\mathbb{R}} f(x)\overline{g(x)}dx \right|.$$

b.
$$\|f\|_p^p = p\int_0^\infty t^{p-1} m(\mathbb{R}(|f|>t))dt.$$

Notations and definitions:

1. $y \downarrow x$ means that $y > x$ and y approaches x; $y \uparrow x$ means that $y < x$ and y approaches x.
2. $X(f > a) := \{x \in X : f(x) > a\}$.
3. 2^X is the power set of X, i.e., the collection of all subsets of X.
4. $M(X)$ is the collection of all measurable functions defined on the measure space $(X, \mathscr{A}, \mu)$. $f \in M^+(X)$ means that $f \in M(X)$ and $f \geq 0$.
5. Let E be a measurable subset of $\mathbb{R}$ with respect to the Lebesgue measure m. $m(E)$ denotes the Lebesgue measure of E.