國立政治大學　110 學年度第二學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	數理統計	考試日期	**2022 年 2 月 21 日**	考試時間	**13:00 至 16:00**

注意事項
- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **4** 個問題, 總計 **100** 分。

1. (20 %) Suppose a random sample $(X_1, \cdots, X_n)$ from the normal distribution with the unknown mean μ and known variance σ^2.

   (a) Find the maximum likelihood estimator (MLE) for $P_\mu(|X_n| \leq k)$.

   (b) Find the uniformly minimum-variance unbiased estimator (UMVUE) for $P_\mu(|X_n| \leq k)$.

2. (20 %) Suppose a random sample $(X_1, \cdots, X_n)$ for the following probability density distribution

$$f_X(x|\theta) = \frac{1}{5 - \theta}, \quad \theta < x < 5$$

   Find the generalized likelihood ratio test (GLRT) for testing $H_0 : \theta = 4$ versus $H_1 : \theta > 4$. Determine the critical region if the significant level $\alpha = 0.05$. In particular, if the sample observations are

$$4.96, 4.64, 4.23, 4.78, 4.90, 4.14, 4.49, 4.54,$$

   do we reject H_0? (Hint: $0.05^{1/8} = 0.688$)

3. (30 %) Let $X_1, \cdots, X_n$ be independent identically distributed observations from the Poisson distribution with mean μ. Define $\theta = e^{-m\mu}$ where $m < n$. We wish to estimate θ using observations of $X_1, \cdots, X_n$. The statistics $Y = \sum_{j=1}^{n} X_j$ is complete sufficient for μ, and hence for θ, you may assume this.

   (a) The simple estimator U takes the value 1 if $X_1 = X_2 = \cdots = X_m = 0$ and value 0 otherwise. Prove that U is unbiased.

   (b) By starting with U or otherwise, show that

$$T = (1 - \frac{m}{n})^Y$$

   is the UMVUE for θ.

   (c) Use the distribution of Y to calculate $E[T^\gamma]$ for $\gamma = 1, 2$ and use the results to show that T is consistent and asymptotically efficient relative to the Cramer-Rao lower bound.

命題紙使用說明：　試題將用原件印製，敬請使用黑色墨水正楷書寫或打字 (紅色不能製版請勿使用)。
Remarks ：　For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.

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4. (30 %) Let $X_1, \cdots, X_n$ be independent identically distributed observations from a density written as

$$f_X(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1$$

with $\theta > 0$.

(a) Find the distributions of $W = -\sum_{i=1}^{n} \log X_i$ and $-2\theta \sum_{i=1}^{n} \log X_i$.

(b) Show that

$$\sqrt{n}\left(\left(\sum_{i=1}^{n} \log X_i\right)^{1/n} - e^{-\theta}\right)$$

converges to a normal distribution in distribution with mean "0" and the particular variance.

(c) Use $-2\theta \sum_{i=1}^{n} \log X_i$ to obtain the $1 - \alpha$ level "shortest-length" confidence interval for θ.

命題老師簽章： **(Teacher's Signature)**	日期： **(Date)**	年　月　日	■ 試題隨卷繳交 ■ 不可使用計算機

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