國立政治大學應用數學系九十學年度第二學期研究生學科考試試題

科目：數理統計

本卷共有6個題目，請選5題作答，每題20分。

1. Let X_1, X_2, X_3 be independent with density $f(x_i) = \exp(-x_i)$, $x_i > 0$. Let $U_1 = X_1+X_2+X_3$, $U_2 = (X_1+X_2)/(X_1+X_2+X_3)$, and $U_3 = X_1/(X_1+X_2+X_3)$. Find the joint density of $\boldsymbol{U} = (U_1, U_2, U_3)$.

2. Suppose that a rat is in a maze with two possible directions. If it chooses left, it wanders around the maze for four minutes and comes back to where it started. If it chooses right, then with probability 1/4 it will depart the maze in five minutes, and with probability 3/4 it will come back to where it started after three minutes. We assume that at the beginning and whenever it returns to the start, it chooses to go right with probability 1/5.

   (a) Find the expected time until the rat escapes the maze.

   (b) Show that the probability that the rat stays in the maze forever is zero.

3. Let $X_1, \ldots, X_n$ be independent, and let $X_i \sim N(\mu, \sigma^2)$, $n > 1$.

   (a) Find the distribution of the sample mean $\overline{X}$.

   (b) Find the distribution of the sample variance S^2.

   (c) Show that $\overline{X}$ and S^2 are independent.

4. Let $X_1, \ldots, X_n$ be independent, and let $X_i \sim N(\mu, \sigma^2)$, $n > 1$. Prove or disprove:

   (a) $\overline{X}$ is the best unbiased estimator of μ.

   (b) S^2 is an efficient estimator of σ^2.

   (c) $\overline{X}S^2$ is an efficient estimator of $\mu\sigma^2$. (Hint: $Var(\overline{X}S^2) = \frac{2\sigma^4}{n(n-1)} + \frac{2\mu^2\sigma^2}{n-1} + \frac{\sigma^6}{n}$.)

5. State and prove the Neyman-Pearson theorem.

6. Let X_1 and X_2 be independent, with $X_i \sim B(1, \theta)$. Consider testing that $\theta = 0.4$ against $\theta > 0.4$. Let $\Phi(X_1, X_2)$ be the nonrandomized test which rejects the null hypothesis if $X_1 = 1$.

   (a) Show that $T = X_1 + X_2$ is a sufficient statistic.

   (b) Show that $\Phi^*(T) = E[\,\Phi(X_1, X_2) \mid T\,] = 0$ if $T = 0$, $\Phi^*(T) = 1/2$ if $T = 1$, and $\Phi^*(T) = 1$ if $T = 2$.