國立政治大學應用數學系九十二學年度第二學期研究生學科考試試題

科目：數理統計

1. Let X and Y be independent, and let $f_1(x) = e^{-x}$, $x > 0$, and $f_2(y) = e^{-y}$, $y > 0$.
   (a) Find the joint density of X and Y.
   (b) Find $P(X + Y \leq 1)$.
   (c) Find $P(X + Y \leq z)$ for all $z > 0$.
   (d) Let $Z = X + Y$. Find the density function of Z.

2. Define two statements A and B as follows:
   A: X and Y are independent;
   B: $E(X|Y = y) = E(X)$.
   (a) State the relationship between statements A and B. (i.e., A $\overset{\longrightarrow}{\longleftarrow}$ B, A $\overset{\longrightarrow}{\not\longleftarrow}$ B, A $\overset{\not\longrightarrow}{\longleftarrow}$ B, or A $\overset{\not\longrightarrow}{\not\longleftarrow}$ B?)
   (b) Prove and/or disprove the relationship given in (a).

3. Let $(X_1, X_2, X_3, X_4) \sim M_4(n, (p_1, p_2, p_3, p_4))$. Let $U = X_2 + X_3$.
   (a) What is the distribution of (X_1, U)?
   (b) Show that the conditional distribution of X_1 given $X_2 = 2$ and $X_3 = 3$ is the same as the conditional distribution of X_1 given $U = 5$.

4. Let $X_i \sim N(i\theta, 1)$ and $X_1, X_2, \ldots, X_n$ are independent.
   (a) Find the MLE for θ and then show that this MLE is a consistent estimator of θ as n approaches infinity.
   (b) Both the MLE and the unbiased estimator of θ^2 are consistent estimators of θ^2.

5. Let $X_1, X_2, \ldots, X_{25}$ be a random sample from a normal distribution with mean μ and variance σ^2 unknown. Find the most powerful size-0.05 test for testing the null hypothesis $H_0 : \mu = \mu_0, \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \mu = \mu_1, \sigma^2 = \sigma_0^2$.

6. Let X_1, X_2, and X_3 be independent, with $X_i \sim B(1, \theta)$. Consider testing that $\theta = 0.3$ against $\theta > 0.3$. Let $\Phi(X_1, X_2, X_3)$ be the nonrandomized test which rejects the null hypothesis if $X_3 = 1$.
   (a) Show that $T = X_1 + X_2 + X_3$ is a sufficient statistic.
   (b) Let $\Phi^*(T) = E[\Phi(X_1, X_2, X_3) \mid T]$. Find $\Phi^*(T)$.