國立政治大學應用數學系九十三學年度第二學期研究生學科考試試題

科目：數理統計

1. Let the conditional density of Y given $\Lambda = \lambda$ be

$$f_1(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!},\ y = 0, 1, 2, \ldots$$

and the density function of Λ be

$$f_2(\lambda) = e^{-\lambda},\ \lambda > 0.$$

   (a) Find the marginal density of Y.

   (b) Find $E(Y|\Lambda = \lambda)$, $E(Y)$.

2. Suppose $X = (X_1, X_2, X_3)$ has joint density function

$$f(x_1, x_2, x_3) = \frac{3}{4\pi},\ x_1^2 + x_2^2 + x_3^2 \le 1.$$

Let $X_1 = U_1 \cos U_2 \sin U_3,\ X_2 = U_1 \sin U_2 \sin U_3,\ X_3 = U_1 \cos U_3,\ 0 \le U_1 \le 1,\ 0 \le U_2 \le 2\pi,$ $0 \le U_3 \le \pi$.

   (a) Find marginal density functions for U_1, U_2, and U_3.

   (b) Find $P(1/4 \le U_1 \le 1/2, \pi/4 \le U_3 \le \pi/2)$.

3. (a) Let (X, Y) be random variables. Show that $E(X) = EE(X|Y)$ and $E(XY) = E(YE(X|Y))$.

   (b) Let (X, Y) be random variables with mean μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. Suppose $E(X|Y) = aY + b$. Show that $b = \mu_X - a\mu_Y$, $a = \rho\sigma_X/\sigma_Y$.

4. Let $X_1, X_2, \ldots, X_n$ be independent with density function $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$.

   (a) Find the MLE of θ.

   (b) Find the best unbiased estimator of $1/\theta$.

   (c) Find an efficient estimator of $1/\theta$.

5. (a) State the following theorems: (i) Information inequality. (ii) Rao-Blackwell theorem.

   (b) Let $X \sim B(10, \theta)$. Consider testing $\theta = 0.3$ against $\theta \ne 0.3$ with the test

$$\Phi(X) = \begin{cases} 1, & \text{if } X > 6 \text{ or } X < 1, \\ 0.4, & \text{if } X = 6, \\ 0.2, & \text{if } X = 1, \\ 0, & \text{if } 1 < X < 6. \end{cases}$$

Find the size of this test.

6. (a) State the following theorems: (i) Chebyshev's inequality (ii) Central limit theorem.

   (b) Let $X_1, X_2, \ldots, X_{10}$ be independent, with $X_i \sim B(1, \theta)$. Find a UMP size-0.05 test that $\theta = 0.4$ against $\theta > 0.4$.

$n = 10$, $p = 0.4$, $\sum_{x=0}^{r} b(x; n, p)$ 值表如下：

r	0	1	2	3	4	5	6	7	8	9	10
	0.0060	0.0464	0.1673	0.3823	0.6331	0.8338	0.9452	0.9877	0.9983	0.9999	1.0000