國立政治大學應用數學系九十四學年度第一學期研究生學科考試試題

科目：數理統計

1. Let (X, Y) be continuous random variables such that $f(x, y) = e^{-y}$, $0 < x < y < \infty$.

   (a) Find $E(Y \mid X = 1)$.

   (b) Find the joint moment generating function of (X, Y).

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the density

$$f(x; \theta) = \theta(1 + x)^{-(1+\theta)}, \ x > 0, \ \theta > 0.$$

   (a) Find the best unbiased estimator of $1/\theta$.

   (b) Find the information lower bound for the variance of an unbiased estimator of $1/\theta$.

3. (a) Let X_1, X_2, X_3 be independent with density

$$f(x_i; \alpha_i) = \frac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i}, \ x_i > 0.$$

   Let $U_1 = X_1 + X_2 + X_3, U_2 = X_2/U_1, U_3 = X_3/U_1$. Find the joint density of (U_1, U_2, U_3).

   (b) Let $X \sim N(0, 1)$ and $\Phi(x) = P(X \le x)$. Find the density of $\Phi(X)$.

4. (a) Suppose that for all n, T_n is an estimator of $\tau(\theta)$. Show that if $bias(T_n) \to 0$, $Var(T_n) \to 0$, then T_n is a consistent sequence of estimators.

   (b) Let $X_n \sim \Gamma(n, 1/n)$. Show that $X_n \xrightarrow{P} 1$.

5. Suppose $X_i \sim N(i\theta, 1)$, $i = 1, 2, 3$, are independent. Find the UMP size-0.05 test for testing $\theta = 2$ against $\theta > 2$. ($\sqrt{14} \doteq 3.741, \Phi(1.645) = 0.95$ where $\Phi(x) = \int_{-\infty}^{x} (1/\sqrt{2\pi}) e^{-t^2/2}\, dt$)

6. Suppose X is a discrete random variable taking on the values $1, 2, 3$, and 4, and θ takes on the values $-1, 0$, and 1. Suppose also that the density of X is giving in the following table:

x	1	2	3	4
$f(x; -1)$	0.53	0.30	0.00	0.17
$f(x; 0)$	0.60	0.20	0.10	0.10
$f(x; 1)$	0.60	0.22	0.18	0

   Consider testing that $\theta = 0$ against $\theta \neq 0$.

   (a) Show that the 0.2 LRT for this problem rejects the null hypothesis if $X = 3$ or 4.

   (b) Show that the test which rejects the null hypothesis if $X = 2$ is a size-0.2 test which is more powerful than the LRT.