國立政治大學應用數學系九十五學年度第一學期研究生學科考試試題

科目：數理統計

1. Let X and Y be independent, and let $f_1(x) = e^{-x}$, $x > 0$, and $f_2(y) = e^{-y}$, $y > 0$. Let $U = X + Y$, $V = X/(X + Y)$. Find
   (a) the joint density of U and V,
   (b) the conditional density of U given $V = v$, and
   (c) the marginal density of U.
2. Let $X_1, \ldots, X_n$ be independent, and let $X_i \sim N(\mu, \sigma^2)$, $n > 1$.
   (a) What are the distributions of the sample mean $\overline{X}$ and the sample variance S^2?
   (b) Are $\overline{X}$ and S^2 independent?
3. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with X_i uniformly distributed on the interval $(0, 1)$. Find the limiting distribution of $W_n = n(1 - \max_{i \le n} X_i)$.
4. Let X_i be independent Poisson distribution with mean $i\theta$, where $i = 1, 2, \ldots, n$. Now, let $P = \sum_{i=1}^{n}(X_i/a_n)$ and $T = P^2 - P/a_n$, where $a_n = \sum_{i=1}^{n} i$. Prove or disprove that
   (a) P is a consistent estimator of θ.
   (b) T is a consistent estimator of θ^2.
5. State and prove the Neyman-Pearson theorem.
6. We say that a statistic T has a complete family of distributions if $E_\theta h(T) = 0$ for all θ implies that $h(T) \equiv 0$. Use this definition to show that $\overline{X}$ has a complete family of distributions and is a complete sufficient statistic, where $X_1, \ldots, X_n$ are independent and $X_i \sim N(\theta, 1)$.