國立政治大學 114學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	數理統計	考試日期	2025 年 9 月 8 日	考試時間	13:00-16:00

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **5** 個問題, 總計 **100** 分。

Please show all your work.

1. (30% Each 10%) Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variable based on the probability density function written as

$$f(x|\theta) = e^{\theta T(x) - \psi(\theta)} h(x).$$

Let the true $\theta = \theta_0 \in \Theta_0$ where Θ_0 is the parameter natural space. Assume $\psi(\theta)$ is infinite differentiable and $\psi''(\theta) > 0$ for all $\theta \in \Theta_0$.

   (a) Show the Fisher information function

$$I(\theta) = \mathbb{E}_\theta \left[\left(\frac{\partial}{\partial \theta} \log f(X|\theta) \right)^2 \right] = \psi''(\theta)$$

   (b) Prove that $\hat{\theta}_{MLE}$ is the consistent estimation of θ_0. Namely, $\hat{\theta}_{MLE} \to \theta_0$ in probablity.

   (c) Prove the asymptotic normality for $\hat{\theta}_{MLE}$. Namely

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta_0) \sim N(0, I^{-1}(\theta_0))$$

   where $I(\theta_0) = \psi''(\theta_0)$.

2. (10%) Let $X_1, \cdots, X_n$ be a random sample from the normal distribution with unknown mean μ and known variance σ^2 written as

$$f(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}.$$

Prove that for testing the simple hypothesis $H_0 : \mu = \mu_0$ versus $H_a : \mu = \mu_1$, the likelihood ration test is the uniformly most powerful (UMP) test. Namely, show the proof of the Neyman Pearson Lemma.

命題老師簽章： **(Teacher's Signature)**	日期：　　年　　月　　日 **(Date)**	■ 試題隨卷繳交 ■ 不可使用計算機

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3. (30% Each 10%) Let $X_1, \cdots, X_n$ be a random sample from the normal distribution with unknown mean μ and known variance σ^2 written as

$$f(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

   (a) For testing $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$, does the uniformly most powerful (UMP) test exist? Please explain in details.

   (b) Given $\alpha = 0.05$, find the likelihood ratio test for testing $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$. Please show the decision function or decision rule based on reject region explicitly.

   (c) Given $\alpha = 0.05$, $\mu_0 = 60$, $n = 100$, $\sigma^2 = 1$, and $\overline{x} = 65$, find the 95% confidence interval. Does the result using likelihood ratio test same as the result using the CI? Please explain in details.

   (Hint: Let Z be the standard normal random variable. We have $\mathbb{P}(Z > 2.362) = 0.01$, $\mathbb{P}(Z > 1.96) = 0.025$, $\mathbb{P}(Z > 1.645) = 0.05$, and $\mathbb{P}(Z > 1.282) = 0.1$)

4. (20% Each 5%) Let $X_1, \cdots, X_n$ be random samples driven by the Bernoulli distribution with the probability of success $0 \leq \theta \leq 1$.

   (a) Find the maximum likelihood estimate.

   (b) Is the maximum likelihood estimator sufficient? Prove or disprove it.

   (c) Is the maximum likelihood estimator consistent? Prove or disprove it.

   (d) Is the maximum likelihood estimator unbiased? Prove or disprove it.

5. (10% Each 5%) Let $X_1, \cdots, X_n$ be random samples driven by the Bernoulli distribution with the probability of success θ. Now, flip coin 10 times with two outcomes $\{H, T\}$. Define H means success. Given the data $\{TTTTTTTTTT\}$, find the maximum likelihood estimate in the following cases with restriction

   (a) $0 < \alpha \leq \theta \leq \beta < 1$

   (b) Given $0 < \gamma_1 < \gamma_2 < \gamma_3 < 1$, we only can choose $\theta = \gamma_1$, $\theta = \gamma_2$, or $\theta = \gamma_3$.

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命題紙使用說明： 試題將用原件印製，敬請使用黑色墨水正楷書寫或打字 (紅色不能製版請勿使用)。
Remarks ： For the convenience of reprinting please Write questions in black or blue-black (but no red) ink.