NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	數理統計	考試日期	2023 年 9 月 18 日	考試時間	13:00 至 16:00

注意事項

- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **6** 個問題, 總計 **120** 分。

Please show all your work.

1. (10 %) Suppose X_1 and X_2 are random variables with joint p.d.f.

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 6(1-x_2), & 0 < x_1 < x_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $U = \dfrac{X_1}{X_2}$ follows a uniform distribution on $(0,1)$.

2. (10 %) Let $X_1, X_2, \cdots, X_n$ denote a random sample from the probability density function

$$f(x) = \begin{cases} \dfrac{2}{x^2}, & x \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Does the weak law of large numbers apply to $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ in this case? Why or why not?

3. (40 %) Let $X_1, X_2, \cdots, X_n$, $n > 2$, denote a random sample with the probability density function

$$f(x) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

where $\theta > 0$.

(a) (10 points) Show that $\bar{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ is a consistent estimator of $\theta/(\theta+1)$.

(b) (10 points) Find the maximum likelihood estimator (MLE) $\hat{\theta}$ of θ.

(c) (10 points) Show that the MLE $\hat{\theta}$ of θ found in (b) is biased?

(d) (10 points) Show that $[(n-1)/n]\hat{\theta}$ is an unbiased but not efficient estimator of θ.

4. (30 %) Let $X_1, X_2, \cdots, X_n$ denote a random sample from the probability density function

$$f(x|\theta) = \begin{cases} \dfrac{3x^2}{\theta^3}, & 0 \leq x \leq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

and let $Y_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$.

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(a) (10 points) Show that $Y_{(n)}$ has probability density function

$$f_{(n)}(x|\theta) = \begin{cases} \dfrac{3nx^{3n-1}}{\theta^{3n}}, & 0 \le x \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (10 points) Show that $Y_{(n)}$ is sufficient for θ.

(c) (10 points) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

5. (10 %) Let $X_1, X_2, \cdots, X_n$ be a random sample from the normal distribution $N(\theta, 1)$. Show that the likelihood ratio test for testing $H_0 : \theta = \theta'$ against $H_1 : \theta \neq \theta'$, where θ' is specified, leads to the inequality of the form $|\bar{x}_n - \theta'| \ge c$, where $\bar{x}_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$.

Some Facts:

Let $X \sim Gamma(a, c)$ and $Y \sim Gamma(b, c)$ be two independent random variables. Then

1. The p.d.f. of X is $f_X(x) = \dfrac{x^{a-1}e^{-\frac{x}{c}}}{c^a\Gamma(a)}$, where Γ is the Gamma function such that $\Gamma(n) = (n-1)!$ for any positive integer n.
2. $E(X) = ac$ and $Var(X) = ac^2$.
3. For any $k > -a$, $E(X^k) = \dfrac{\Gamma(a+k) \cdot c^k}{\Gamma(a)}$.
4. $X + Y \sim Gamma(a+b, c)$.